

Twin Primes and the Zeros of the Riemann Zeta Function

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October 3, 2012

Abstract

The Legendre type relation for the counting function of ordinary twin primes is reworked in terms of the inverse of the Riemann zeta function. Its analysis sheds light on the distribution of the zeros of the Riemann zeta function in the critical strip and their links to primes and the twin prime problem.

MSC: 11A41, 11N05, 11M06

Keywords: zeta zeros, twin primes, twin ranks, non-ranks, remnants

1 Introduction

The pair sieve for ordinary twin primes [1] leads to a formula for the twin prime counting function $\pi_2(x)$ that is analogous to Legendre's formula [2] for the prime number counting function $\pi(x)$. Before and after separating it into main and error terms [1], it is rewritten here using relevant Dirichlet series. Since the Riemann zeta function ends up in the denominator of the contour integrals, this feature links the zeta zeros to twin primes, much like $\pi(x)$ or related counting functions are expressed as Perron integrals over

ζ'/ζ in analytic number theory [3],[4]. Our analysis sheds light on the role of twin primes in the distribution of the nontrivial zeros of the Riemann zeta function, which are those in the critical strip, as usual.

In Sect. 2 the main concepts, such as twin ranks, non-ranks and remnants of the twin-prime pair sieve are recalled along with its main result, the Legendre type formula for π_2 . In Sect. 3 it is rewritten as a Perron integral and analyzed. In Sect. 4 the findings are summarized and discussed.

2 Review of the Pair Sieve and Notations

The prime numbers 2, 3 do not play an active role here because they are not of the standard form $6m \pm 1$. This also applies to the first twin prime pair 3, 5. From now on p denotes a prime number and p_j the j th prime with $p_1 = 2, p_2 = 3, p_3 = 5, \dots$

Definition 2.1. If $6m \pm 1$ is an ordinary twin prime pair for some positive integer m , then m is its *twin rank*. A positive integer n is a *non-rank* if $6n \pm 1$ are **not both** prime.

The arithmetical function $N(x)$, $x \neq n + \frac{1}{2}$ is needed for non-ranks.

Definition 2.2. If x is real then $N(x)$ is the integer nearest to x . The ambiguity for $x = n + \frac{1}{2}$ with integral n will not arise.

In Ref. [1] we then prove

Lemma 2.3. *If $p \geq 5$ is prime then the positive integers*

$$k(n, p)^\pm = np \pm N\left(\frac{p}{6}\right) > 0, \quad n = 0, 1, 2, \dots \quad (1)$$

are non-ranks. If an integer $k > 0$ is a non-rank, there is a prime $p \geq 5$ so that Eq. (1) holds with either + or - sign.

This means that the pairs $6k^+ \pm 1$ and $6k^- \pm 1$ each contain at least one composite number. Therefore, the primes $p \geq 5$ organize all non-rank numbers in pairs of arithmetic progressions. These pairs are twin prime analogs of multiples np , $n > 1$, of primes struck from the integers in Eratosthenes' sieve.

Given a prime $p \geq 5$, when all non-ranks to primes $5 \leq p' < p$ are subtracted from the non-ranks to p , then the non-ranks to **parent** prime p are left forming the set \mathcal{A}_p . This process [1] naturally introduces the primorial $L(p) = \prod_{5 \leq p' < p} p'$ as the period (of its arithmetic progressions).

Definition 2.4. Let $p \geq p' \geq 5$ be prime. The supergroup $\mathcal{S}_p = \bigcup_{p' \leq p} \mathcal{A}_{p'}$ contains the sets of arithmetic non-rank progressions of all $\mathcal{A}_{p'}$, $5 \leq p' \leq p$.

The number $S(p)$ counts the non-ranks of \mathcal{S}_p over one period $L(p)$.

Definition 2.5. Since [1] $L(p) > S(p)$, there is a set \mathcal{R}_p of *remnants* r in its first period such that $r \notin \mathcal{S}_p$; they are twin-ranks or non-ranks to primes $p_j < p < L(p)$.

3 Reworking the Twin Prime Formula

If p_j is the j th prime, then we need

$$\begin{aligned} L(p_j) &= \prod_{5 \leq p \leq p_j} p, \quad M(j+1) = \frac{1}{6}(p_{j+1}^2 - 1), \\ x &= L(p_j) - M(j+1), \quad y = 6(x + M(j+1)) + 1 \end{aligned} \quad (2)$$

for the main result of Ref. [1], which is a Legendre type formula for the number R of twin ranks in the first period of length $L(p_j)$ of the supergroup \mathcal{S}_{p_j} and $\pi_2(y)$ counting twin pairs below y :

$$R = \pi_2(y) - 1 = R_0 + \sum_{(p_j) < n \leq x} \mu(n) 2^{\nu(n)} \left\lfloor \frac{x}{n} \right\rfloor, \quad (3)$$

where $[z]$ is the greatest integer function, and

$$R_0 = L(p_j) \prod_{5 \leq p \leq p_j} \left(1 - \frac{2}{p}\right) \sim \frac{Cx}{(\log \log x)^2}, \quad p_j \sim \log x \rightarrow \infty \quad (4)$$

counts the number of remnants in \mathcal{S}_{p_j} , that is, twin ranks **and** non-ranks to primes $p_j < p < x$. Therefore, the n in the \sum_n of Eq. (3) run over these primes only and their products, and the upper limit is x because the greatest integer function $\left\lfloor \frac{x}{n} \right\rfloor = 0$ for $n > x$.

The Dirichlet series characteristic of twin primes and associated with R_0 are

$$P_j(s) = \prod_{p > p_j} \left(1 - \frac{2}{p^s}\right) = \prod_{p \leq p_j} \left(1 - \frac{2}{p^s}\right)^{-1} \sum_{n=1}^{\infty} \mu(n) 2^{\nu(n)} n^{-s}. \quad (5)$$

They converge absolutely for $\sigma > 1$, as is evident from the majorant [5]

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\nu(n)}}{n^s}, \quad \sigma > 1. \quad (6)$$

Note that $2^{\nu(n)} \sim \log n / \zeta(2)$ in the interval $[1, x]$ on average, which is shown in 4.4.18 of Ref. [3]. The corresponding Dirichlet series for primes is

$$P_0(s) = \prod_{p \geq 2} \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)}, \quad (7)$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann zeta function.

We now use the Perron formula in essentially the forms proved in 4.4.15 and 4.4.16 of Ref. [3].

Lemma 3.1. *Let the Dirichlet series $A(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be absolutely convergent for $\sigma = \Re(s) > 1$. (i) Then*

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} A(s) \frac{x^s}{s} ds + O\left(\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^{\sigma} |a_n| \min\left(1, \frac{1}{T|\log \frac{x}{n}|}\right)\right). \quad (8)$$

where the lhs $\sum_{n \leq x}$ means that for $n = x$, a_n is reduced by $1/2$.

(ii) If $a_n = O(n^{\varepsilon})$ for any $\varepsilon > 0$, then

$$\sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} A(s) \frac{x^s}{s} ds + O\left(\frac{x^{\sigma+\varepsilon}}{T}\right). \quad (9)$$

Corollary 3.2. (i) For $\sigma > 1$

$$\begin{aligned} \sum_{n \leq x} a_n \left[\frac{x}{n}\right] &= \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} A(s) \zeta(s) \frac{x^s}{s} ds \\ &+ O\left(\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^{\sigma} \sum_{d|n} |a_d| \min\left(1, \frac{1}{T|\log \frac{x}{n}|}\right)\right). \end{aligned} \quad (10)$$

(ii) If $a_n = O(n^{\varepsilon/2})$ for any $\varepsilon > 0$ then

$$\sum_{n \leq x} a_n \left[\frac{x}{n}\right] = \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} A(s) \zeta(s) \frac{x^s}{s} ds + O\left(\frac{x^{\sigma+\varepsilon}}{T}\right). \quad (11)$$

Proof. (i) This follows from (i) in Lemma 3.1 and the proof of 4.4.15 in Ref. [3] using

$$\sum_{N \leq x} \sum_{n|N} a_n = \sum_{n \leq x} a_n \left[\frac{x}{n}\right], \quad A(s) \zeta(s) = \sum_{N=1}^{\infty} \frac{1}{N^s} \sum_{n|N} a_n. \quad (12)$$

For (ii) we use $\sum_{d|n} |a_d| \ll n^{\varepsilon/2} d(n) \ll n^\varepsilon$, since $d(n) = \sum_{d|n} 1 = O(n^{\varepsilon/2})$ is well known to hold [4], in conjunction with the proof of 4.4.16 in Ref. [3]. \diamond

Lemma 3.3.

$$P_1(s)\zeta^2(s) = (1 - \frac{1}{2^s})^{-2} \prod_{p>2} \left(1 + \frac{1}{p^s(p^s - 2)}\right)^{-1} = \frac{(1 - 2^{-s})^{-2}}{D(s)} \quad (13)$$

$$D(s) = \prod_{p>2} \left(1 + \sum_{\nu=0}^{\infty} \frac{2^\nu}{p^{(\nu+2)s}}\right) = 1 + \sum_{N=4}^{\infty} \frac{2^{2r_e(N)+2r_o(N)-2\bar{r}_e(N)-2\bar{r}_o(N)}}{N^s} \quad (14)$$

converges absolutely for $\sigma > 1/2$. Here

$$r_e(N) = \sum_{i=1}^m \nu_i, \quad r_o(N) = \sum_{i=1}^n (\mu_i + 3), \quad \bar{r}_e(N) = \sum_{\nu_i>0} 1, \quad \bar{r}_o(N) = \sum_{\mu_i>0} 1 \quad (15)$$

are additive functions for

$$N = p_{e_1}^{2(\nu_1+1)} \dots p_{e_m}^{2(\nu_m+1)} p_{o_1}^{2\mu_1+3} \dots p_{o_n}^{2\mu_n+3} \quad (16)$$

in Eq. (14).

Proof. Substituting in

$$\prod_{p>2} \frac{(1 - \frac{2}{p^s})}{(1 - \frac{1}{p^s})^2} = \frac{1}{\prod_{p>2} (1 + \frac{1}{p^s(p^s-2)})} \quad (17)$$

the expansions

$$\frac{1}{1 - \frac{2}{p^s}} = 1 + \frac{2}{p^s} + \frac{2^2}{p^{2s}} + \dots, \quad (18)$$

$$1 + \frac{1}{p^{2s}(1 - \frac{2}{p^s})} = 1 + \sum_{\nu=0}^{\infty} \frac{2^\nu}{p^{(\nu+2)s}}, \quad (19)$$

yields Eq. (14) with N of the form in Eq. (16). \diamond

Thus for $\sigma > 1$

$$\begin{aligned} P_j(s)^{-1} &= \zeta^2(s) (1 - \frac{1}{2^s})^2 \prod_{2 < p \leq p_j} (1 - \frac{2}{p^s}) \prod_{p>2} \left(1 + \frac{1}{p^s(p^s - 2)}\right) \\ &= \left(\frac{P_1(s)}{\prod_{2 < p \leq p_j} (1 - \frac{2}{p^s})} \right)^{-1} \end{aligned} \quad (20)$$

with $P_1(s)$ from Eq. (13).

We now apply (ii) of Cor. 3.2 to $P_j(s)$. This yields the Legendre type formula **before** it is split into its main and error terms according to Ref. [1] so that the leading asymptotic term is R_0 .

Theorem 3.4. For $\sigma > 1$, $R = \frac{1}{2}(\pi_2(y) - 2)$, $R_0 = L(p_j) \prod_{5 \leq p \leq p_j} (1 - \frac{2}{p})$ and x, y from Eq. (2),

$$R = R_0 + \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \frac{(1 - \frac{1}{2s})^{-2} x^s ds}{s\zeta(s) \prod_{2 < p \leq p_j} (1 - \frac{2}{p^s}) D(s)} + O\left(\frac{x^{\sigma+\bar{\varepsilon}}}{T}\right), \quad (21)$$

with $D(s)$ from Eq. (14) and any $\bar{\varepsilon} > 0$.

Proof. We replace in Eq. (3) the remainder sum by the Perron integral of Cor. 3.2 with $A(s) = P_j(s)$ using Lemma 3.3 for $P_1(s)$ in conjunction with Eq. (20). Canceling the factor $\zeta(s)$, this yields the Perron integral in Eq. (21).

The zeros $s_p = \log 2 / \log p$ of $\prod_{p \leq p_j} (1 - 2/p^s)$ cancel the corresponding poles in $D(s)$. The Euler product of $D(s)$ in Eq. (13) guarantees no zeros for $\sigma > 1/2$. Note that

$$\sum_{d|n} |\mu(d)| 2^{\nu(d)} \leq 2^{\nu(\tilde{n})} d(\tilde{n}), \quad d(\tilde{n}) = \sum_{d|\tilde{n}} 1, \quad (22)$$

where \tilde{n} is the product of different prime divisors of n . Moreover,

$$d(\tilde{n}) = 2^{\nu(\tilde{n})}, \quad d(\tilde{n}) = O(\tilde{n}^{\bar{\varepsilon}/2}) = O(n^{\bar{\varepsilon}/2}), \quad (23)$$

where the last bound is well known [4]. Thus, (ii) in Cor. 3.2 applies, which completes the proof. \diamond

Corollary 3.5. The Riemann hypothesis (RH) is incompatible with the twin prime formula (21) of Theor. 3.4.

Proof. Assuming RH, we shift the line of integration in Eq. (21) from $\sigma > 1$ to $\sigma = \frac{1}{2} + \bar{\varepsilon}$ for any $\bar{\varepsilon} > 0$ using Cauchy's theorem. Since RH implies the Lindelöf hypothesis [5], we know that

$$\frac{1}{\zeta(s)} = O(|t|^\delta), \quad s = \sigma + it, \quad \sigma \geq \frac{1}{2} + \bar{\varepsilon} \quad (24)$$

for any $\delta > 0$ that may depend on $\bar{\varepsilon}$. On $\sigma = \frac{1}{2} + \bar{\varepsilon}$ the Perron integral in Theor. 3.4 obeys

$$\int_{\frac{1}{2}+\bar{\varepsilon}-iT}^{\frac{1}{2}+\bar{\varepsilon}+iT} \frac{(1 - \frac{1}{2s})^{-2} x^s ds}{s\zeta(s) \prod_{2 < p \leq p_j} (1 - \frac{2}{p^s}) D(s)} = O(T^\delta x^{\frac{1}{2}+\bar{\varepsilon}} \log T (\log \log x)^2), \quad (25)$$

because $|D(s)|^{-1} = O(1)$, $\prod_{5 \leq p \leq p_j} |1 - \frac{2}{p^s}|^{-1} = O((\log \log x)^2)$ with $p_j = O(\log x)$. On the horizontal line segments from $\frac{1}{2} + \tilde{\varepsilon} \pm iT$ to $\sigma \pm iT$ the Perron integral is bounded by $O(T^{\delta-1} x^\sigma (\log \log x)^2 / \log x)$. Taking $T = x^\alpha$ and equating the exponents of x in both error terms in Eqs. (21),(25) determines

$$\alpha = \frac{\frac{1}{2} + \varepsilon + \bar{\varepsilon} - \tilde{\varepsilon}}{1 + \delta} \approx \frac{1}{2}. \quad (26)$$

Therefore the Perron integral plus error terms in Eq. (21) are $O(x^\alpha)$ and $O(x^\beta)$ with

$$\beta = 1 - \frac{1}{2} \frac{1 - \delta}{1 + \delta} + \frac{2\varepsilon\delta}{1 + \delta} + (\tilde{\varepsilon} - \bar{\varepsilon}) \frac{1 - \delta}{1 + \delta} \approx \frac{1}{2} \quad (27)$$

and cannot reduce $R_0 \sim Cx/(\log \log x)^2$ to the known bound $O(x/(\log x)^2)$ for π_2 . \diamond

We address next the remainder of the twin prime formula (3), after extracting its asymptotic law [1], using the following Perron integral.

Corollary 3.6. *Let $A(s)$ be absolutely convergent for $\sigma > 1$, and $a_n = O(n^{\varepsilon/2})$, then for $\sigma > 1$*

$$\sum_{n < x} a_n \left\{ \frac{x}{n} \right\} = \frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} ds x^s A(s) \left[\frac{1}{s - 1} - \frac{\zeta(s)}{s} \right] + O\left(\frac{x^{\sigma + \varepsilon}}{T} \right). \quad (28)$$

Proof. Using

$$\left\{ \frac{x}{n} \right\} = \frac{x}{n} - \left[\frac{x}{n} \right] \quad (29)$$

and applying (ii) of Lemma 3.1 to $xA(s+1)$, integrated along the line $\sigma > 0$, and (ii) of Cor. 3.2 we obtain the Perron integral in Eq. (28) upon shifting $s \rightarrow s - 1$ in the first term. The first error term $xO(\frac{x^{\sigma-1+\varepsilon}}{T})$ combines with the second one to that in Eq. (28). \diamond

We now apply Cor. 3.6 to $P_j(s)$ which yields the Perron integral for the error term R_E of Ref. [1] after separating the Legendre type formula into its main and error terms so that the main term obeys the proper asymptotic law expected for twin primes [1] realizing the replacement of $\log \log x \rightarrow \log x$ asymptotics in R_0 .

Theorem 3.7. *For any $\varepsilon > 0$ and $\sigma = 1 + \varepsilon$, $c > 0$ the twin prime remainder*

$$\begin{aligned}
-R_E &= \sum_{(p_j) < n < x} \mu(n) 2^{\nu(n)} \left\{ \frac{x}{n} \right\} = O\left(\frac{x^{1+\varepsilon}}{T}\right) + O\left(\frac{x^{1+\varepsilon} \log^{14} T}{T \log x} (\log \log x)^2\right) \\
&+ O\left(x^\delta \log^{15} T (\log \log x)^2\right) \\
&= O\left(x (\log x)^{15/8} \exp(-c^{1/8} (\log x)^{1/8}) (\log \log x)^2\right). \tag{30}
\end{aligned}$$

Proof. We start from Cor. 3.6 for $P_j(s)$:

$$\begin{aligned}
-R_E &= \sum_{(p_j) < n < x} \mu(n) 2^{\nu(n)} \left\{ \frac{x}{n} \right\} \\
&= \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \frac{(1 - \frac{1}{2^s})^{-2} x^s ds}{\zeta(s) \prod_{2 < p \leq p_j} (1 - \frac{2}{p^s}) D(s)} \left[\frac{1}{(s-1)\zeta(s)} - \frac{1}{s} \right] \\
&+ O\left(\frac{x^{1+\varepsilon}}{T}\right) \tag{31}
\end{aligned}$$

By Chapt. 42, p. 593 of Ref. [6] we have

$$\frac{1}{|\zeta(s)|} = O(\log^5 T), \quad 3 \leq |t| \leq T, \quad \delta \leq \sigma \leq 1 + \varepsilon, \quad \delta = 1 - \frac{c}{\log^7 T} \tag{32}$$

and $\frac{1}{|\zeta(s)|} = O(1)$ for $|t| \leq 3$, $\sigma \geq \delta$. Let R_T be the rectangle joining the vertices

$$1 + \varepsilon - iT, \quad 1 + \varepsilon + iT, \quad \delta + iT, \quad \delta - iT. \tag{33}$$

Then the bounds of $|\zeta(s)|^{-1}$ in Eq. (32) and below hold on the boundary of R_T . By Cauchy's theorem we have

$$\int_{R_T} ds \frac{(1 - \frac{1}{2^s})^{-2} x^s}{\zeta(s) \prod_{2 < p \leq p_j} (1 - \frac{2}{p^s}) D(s)} \left(\frac{1}{(s-1)\zeta(s)} - \frac{1}{s} \right) = 0. \tag{34}$$

The bound for $|\zeta(s)|^{-1}$ implies the bound

$$\frac{1}{2\pi i} \left(\int_{1+\varepsilon+iT}^{\delta+iT} + \int_{\delta-iT}^{1+\varepsilon-iT} \right) (\cdots) = O\left(\frac{x^{1+\varepsilon} \log^{14} T}{T \log x} (\log \log x)^2\right) \tag{35}$$

for the integrals along the horizontal segments upon using $ds = d\sigma$ in

$$\int_{\delta}^{1+\varepsilon} x^{\sigma} d\sigma = \frac{x^{\sigma}}{\log x} \Big|_{\delta}^{1+\varepsilon}. \quad (36)$$

The bound for the vertical integral

$$\frac{1}{2\pi i} \left(\int_{\delta-iT}^{\delta+iT} \right) (\cdots) = O(x^{\delta} \log^{15} T (\log \log x)^2). \quad (37)$$

Putting all this together we obtain the middle section of Eq. (30). Now equating error terms using $T = \exp(c_1 \log^{\alpha} x)$ yields

$$\alpha = \frac{1}{8}, \quad c_1 = c^{1/8}. \quad (38)$$

Substituting α, c_1 into the middle section of Eq. (30) proves the error term on the rhs of Eq. (30). \diamond

This proves that the asymptotic law obtained in Ref. [1] is valid with the remainder smaller than it by any positive power of $\log x$.

4 Summary and Discussion

When the Legendre type formula for π_2 is reworked into a Perron integral involving $\zeta^{-1}(s)$, the nontrivial zeta zeros are seen to be linked to the twin prime counting function π_2 . The asymptotic law of its leading term

$$R_0 = L(p_j) \prod_{5 \leq p \leq p_j} \left(1 - \frac{2}{p}\right) \sim \frac{Cx}{(\log \log x)^2}, \quad x \rightarrow \infty \quad (39)$$

with $x = L(p_j) - M(j+1)$, $p_j \sim \log L(p_j) \sim \log x$ requires $\log \log x$ to become large. In contrast, only $\log x$ is large in the prime number theorem [3],[4]. Therefore, the true asymptotic region of twin primes starts much higher up than for primes. Present numerical results of twin primes and nontrivial zeta zeros have not yet reached the asymptotic twin prime realm. This is valid whether or not there are infinitely many twin primes, because R_0 is the number of remnants including twin pairs (i.e. twin ranks) **and** non-ranks to primes $p_j < p < L(p_j)$. The Perron integral represents the latter's contributions that will reduce R_0 to $R = \pi_2(y) - 1$ with $y = 6(x + M(j +$

1)) + 1, since R_0 is much larger than known bounds from sieve theory [3],[7] on $\pi_2(y)$ that are due to V. Brun, A. Selberg and others. Only nontrivial zeta zeros in the Perron integral can produce terms that reduce R_0 to the proper magnitude. Our first result is that the zeros on the critical line cannot do the job. Despite trillions of initial zeros on the critical line that are relevant for the prime number distribution without asymptotic twin prime attributes, once twin prime asymptotics matter zeta zeros must move off the critical line toward the borders of the critical strip.

Finally, from the point of view of our twin prime formulas (3),(21) a finite number of twin primes is neither a simple nor natural case, as it would require the cancellation of the leading and all subleading asymptotic terms involving fine-tuning of the large primes ($> p_j$) that organize the non-ranks.

But this never happens because, when the Perron integral is developed for $\sum_{n < x} \mu(n) 2^{\nu(n)} \{x/n\}$ and then applied to $P_j(s)$ in the known zero-free region of the Riemann zeta function, the twin prime theorem near primorial arguments follows, hence our second result.

Third, this analysis extends—mutatis mutandem— to all other twin prime cases [8],[9],[10] of the classes I, II, III of the classification [11],[12], that is to say that any twin prime case prevents RH from being valid, except for the initial long stretch, and has an asymptotic law of the expected form at primorial arguments.

References

- [1] H. J. Weber, Twin Prime Sieve, www.arxiv.org/1203.5240, UVa preprint.
- [2] H. Riesel, *Prime Numbers and Computer Methods for Factorization*, 2nd ed., Birkhäuser, Boston, 1994.
- [3] M. Ram Murty, *Problems in Analytic Number Theory*, Springer, New York, 2001.
- [4] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 5th ed., 1988.
- [5] E. C. Titchmarsh and D. R. Heath-Brown, *The Theory of the Riemann Zeta-Function*, 2nd ed., Oxford, Clarendon Press, (1986).

- [6] E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Teubner, Leipzig, 1909.
- [7] R. Friedlander and H. Iwaniec, *Opera Cribro*, Amer. Math. Soc. Colloq. Publ. **57**, Prov. RI, (2010).
- [8] H. J. Weber, Sieves for Twin Primes in Class I, www.arxiv.org/1204.5728, UVa preprint.
- [9] H. J. Weber, Sieves for Twin Primes in Class II, UVa preprint.
- [10] H. J. Weber, Sieves for Twin Primes in Class III, UVa preprint.
- [11] H. J. Weber, Regularities of Prime Number Twins, Triplets and Multiplets, Global. J. Pure Applied Math. **9** (2013), www.adsabs.harvard.edu/abs/2011arXiv1103.0447W.
- [12] H. J. Weber, Exceptional Prime Number Twins, Triplets and Multiplets, www.arxiv.org/1102.3075, UVa preprint.